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A wavelet-Galerkin method for inhomogeneous diffusion equations subject to mass specification

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Abstract
A wavelet-Galerkin procedure is derived and implemented for the numerical solution of inhomogeneous diffusion equations subject to mass specification involving non-polynomial functions. It is demonstrated that the accuracy of approximation of these functions by polynomial interpolation at Chebyshev nodes is in good agreement with the exact solution for several numerical experiments.

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1. Introduction

Several physical problems can be modelled by the following partial differential equations:

$$u_t(x,t) = u_{xx}(x,t) + S(x,t) \qquad 0 < x < 1 \qquad 0 < t \leqslant T \tag{1}$$

$$u(x,0) = f(x) \qquad 0 < x < 1 \tag{2}$$

$$u(1,t) = g(t) \qquad 0 < t \leqslant T \tag{3}$$

$$\int_0^\beta u(x,t)\,\mathrm{d}x = m(t) \qquad 0 < \beta < 1. \tag{4}$$

where $S(x,t)$, $f(x)$, $g(t)$ and $m(t)$ are known functions (assumed to be sufficiently smooth to guarantee a unique smooth solution) and T is a given positive constant. The subscripts x and t denote the respective partial differentiation.

The existence, uniqueness and continuous dependence on data of the solution to this problem have been studied by Cannon *et al* [1–3]. Some numerical schemes (other than the wavelet-Galerkin scheme) for solving problems of this sort have been given in Cannon *et al* [4, 5] and Fairweather and Saylor [6].

Currently, several investigations are being carried out which aim at exploiting the properties of wavelet functions in widely different areas. In areas such as time-series analysis, image processing, approximation theory and preconditioning dense matrices, wavelets are recognized as a powerful tool. Another area in which wavelet analysis is gaining considerable attention is the numerical analysis of differential equations. After discretizing the differential equation in a conventional way like the finite difference approximation, wavelets can be used for algebraic manipulations in the system of equations obtained which may lead to a better condition number on the resulting system. An approach to the study of partial differential equations is based on this idea where it has been shown that the differential operator is approximately diagonal in the wavelet bases. Another approach to the study of differential equations is to use the wavelet bases in place of other conventional bases such as Fourier, Legendre or Chebyshev bases in the spectral methods. The purpose of the present paper is to implement the wavelet-Galerkin method for the problem described by equations (1)–(4). In this direction, we will adopt the application of wavelet bases (precisely the scaling function bases) to approximate the solution of this problem.

Generally, applying the wavelet-Galerkin procedure to the differential equations involves the evaluation of connection coefficients to approximate derivatives. The connection coefficients are integrals with integrands being the product of scaling functions and their derivatives (see for example Beylkin [7] and Chen *et al* [8]).

The main obstacle in that direction is that these connection coefficients require the inhomogeneity term $S(x,t)$ in the differential equation as well as the function $f(x)$ of the initial condition to be in polynomial form. Therefore, most works that use wavelet-Galerkin methods to solve partial differential equations are limited to cases in which the inhomogeneity term and initial conditions are polynomials in the spatial variable (see Chen *et al* [8], Lin and Zhou [9] and Dahmen [10]). However, in the present work we overcome this difficulty by interpolating the non-polynomial functions $S(x,t)$ and $f(x)$ in terms of suitable polynomials in the spatial variable x. It has been shown that the accuracy of approximation of these functions by polynomial interpolation at Chebyshev nodes is in good agreement with the exact solution for several numerical experiments. A sample of these numerical experiments is exhibited in this paper.

2. Preliminaries and fundamentals

This section is devoted to introducing some basic properties of wavelets, an error estimate for expanding a function by scaling function bases, a definition of some connection coefficients and an error estimate for approximating functions by interpolating polynomials.

Firstly, we give a brief review of the constructions and some basic properties of wavelets. Daubechies [11] constructed a family of orthonormal bases of compactly supported wavelets for the space of square-integrable functions $L^2(\mathbb{R})$. The wavelet function $\varpi(x)$ is derived from the normalized scaling function $\varphi(x)$ described by the scaling equation

$$\varphi(x) = \sum_{k=0}^{L-1} h_k \varphi(2x-k)$$

where $h_k, k = 0, \ldots, L-1$ is a set of N (an even integer) coefficients that were computed by constructing a certain trigonometric polynomial, see Daubechies [12]. Once $\varphi(x)$ is determined, the multiresolution approach asserts that

$$\varpi(x) = \sum_{k=0}^{L-1} (-1)^k h_{N-k-1} \varphi(2x-k)$$

will be a wavelet. Define, for $j, k \in \mathbb{Z}$ (the set of integer numbers)

$$\varphi_{j,k}(x) = \sqrt{2^j}\varphi(2^j x - k) \tag{5}$$

and

$$\varpi_{j,k}(x) = \sqrt{2^j}\varpi(2^j x - k).$$

Let $W_j = \text{span}\,\{\varpi_{j,k}(x) : k \in \mathbb{Z}\}$. Then $\bigoplus_{j=-\infty}^{\infty} W_j = L^2(\mathbb{R})$. Let $V_j = \bigoplus_{l=-\infty}^{j} W_l$. Since it is not popular to deal with both infinities, we usually let $V_J = V_{-1} \bigoplus_{j=0}^{J} V_j$, $J \geqslant 0$.

Numerical computations are carried out for a sufficient large J. If $\varpi(x)$ is related to a multiresolution approximation of L^2, then there is a scaling function $\varphi(x)$ such that $\{\varphi_{j,k}(x) : k \in \mathbb{Z}\}$ forms an orthonormal basis for V_j. Thus there are two kinds of bases for V_J. The first involves both φ and ϖ, and the second involves φ only. A function may be expanding in both kinds of base. For more detailed discussions one may consult Strang [13] and Chui [14].

Approximation properties of the finite-dimensional subspaces from which we choose the trail functions is a crucial part of the error analysis in Galerkin methods. For the family of Daubechies wavelets, Xu and Shann [15] proved that for a fixed t, $u(x, \cdot) \in H^s(\Omega)$, $0 \leqslant s \leqslant \frac{L}{2}$ and $P_J u = \sum_k u_{J,k}\varphi_{J,k}(x)$ being the L^2 projection of $u(x, \cdot)$ into V_J, we have

$$\|u - P_J u\| \leqslant \gamma 2^{-sJ}|u|_s \tag{6}$$

where γ is some positive constant independent of J and u. Here $H^s(\Omega)$ is the standard Sobolev space and $|u|_s$ is the semi-norm defined on this space, i.e.

$$|u|_s^2 = \int_\Omega |u^{(s)}(x, \cdot)|^2 \,\mathrm{d}x \qquad \Omega = (a, b) \quad (-\infty < a < b < \infty).$$

Connection coefficients (involving the scaling function) always occur in the application of the wavelet-Galerkin procedure to differential equations. The connection coefficients required for the problem at hand are

$$\theta_i(x) = \int_0^x \int_0^{y_n} \cdots \int_0^{y_2} \varphi(y_1)\,\mathrm{d}y_1 \cdots \mathrm{d}y_{i-1}\,\mathrm{d}y_i \tag{7}$$

$$M_k^i(x) = \int_0^x y^i \varphi(y - k)\,\mathrm{d}y \tag{8}$$

and

$$\Gamma_k^n(x) = \int_0^x \varphi^{(n)}(y - k)\varphi(y)\,\mathrm{d}y \tag{9}$$

where $\varphi^{(n)}(x)$ denotes the nth derivative of $\varphi(x)$. Algorithms for computing these coefficients are given in Chen *et al* [8]. When a set of connection coefficients required for a class of equations is computed and stored, it may result in computational advantages. We computed these three connection coefficients utilizing a Matlab coding based on algorithms in Chen *et al* [8].

Before concluding this section let us refer to an error estimate of approximating a continuously differentiable function by a polynomial interpolation at Chebyshev zeros. Let $p_n(x)$ be the polynomial of degree at most n interpolating to $Y(x)$ at the zeros of $(n + 1)$ Chebyshev polynomial. If $Y(x)$ is $(n + 1)$-times continuously differentiable, then

$$|Y(x) - p_n(x)| \leqslant \frac{1}{2^n(n + 1)!} \max_{t \in \Omega} |Y^{(n+1)}(t)| \tag{10}$$

where

$$p_n(x) = \sum_{i=0}^{n} \alpha_i x^i. \tag{11}$$

The coefficients α_i are to be determined utilizing $Y(x)$ (see for instance Asithambi [16] p 428).

3. Main results

In the following, we apply the proposed wavelet-Galerkin approximation scheme to find the solution of the inhomogeneous diffusion equation subject to mass specification (1)–(4).

Let $u(x, t)$ be approximated as an expansion of scaling functions at level J by

$$u(x, t) \simeq \sum_{k=2-L}^{2^J-1} u_{J,k}(t)\varphi_{J,k}(x) \tag{12}$$

where $J \geqslant 1$ and $\varphi_{J,k}(x)$ is given by (5). Substituting (12) into (1) and then applying the Galerkin discretization scheme yields

$$\begin{aligned} \sum_{k=2-L}^{2^J-1} a_{l,k} \frac{\mathrm{d}}{\mathrm{d}t} u_{J,k}(t) &= \sum_{k=2-L}^{2^J-1} b_{l,k} u_{J,k}(t) + 2^{J/2} \int_0^1 S(x, t)\varphi(2^J x - l)\,\mathrm{d}x \\ l &= 2 - L, 3 - L, \ldots, 2^J - 1. \end{aligned} \tag{13}$$

The coefficients $a_{l,k}$ and $b_{l,k}$ in the above system of first-order differential equations are given by

$$a_{l,k} = \Gamma^0_{k-l}(2^J - l) - \Gamma^0_{k-l}(-l) \tag{14}$$

$$b_{l,k} = 2^{2J} \left\{ \Gamma^2_{k-l}(2^J - l) - \Gamma^2_{k-l}(-l) \right\} \tag{15}$$

where $\Gamma^n_k(x)$ is defined in (9). To compute the integrals on the right-hand side of (13), assume that $S(x, t) = S_1(x) \cdot S_2(t)$ and $S_1(x)$ is $(n + 1)$-times continuously differentiable. Hence $S_1(x)$ can be approximated by a polynomial as in (11) where its coefficients, say α_{1i}, are determined using $S_1(x)$. The integrals on the right-hand side of system (13), denoted by e_l, take the form

$$e_l = \sqrt{2^J} S_2(t) \sum_{i=0}^{n} 2^{-(i+1)J} \alpha_{1i} M^i_l(2^J) \qquad l = 2 - L, 3 - L, \ldots, 2^J - 1 \tag{16}$$

where $M^i_l(2^J)$ is defined by (8).

The initial conditions for the system of differential equations (13) are derived from the initial condition $u(x, 0)$ of the problem at hand by substituting series (12) into condition (2) and applying the Galerkin discretization scheme. The initial conditions $u_{J,k}(0)$ satisfy

$$\sum_{k=2-L}^{2^J-1} a_{l,k} u_{J,k}(0) = 2^{J/2} \int_0^1 f(x)\varphi(2^J x - l)\,\mathrm{d}x \qquad l = 2 - L, 3 - L, \ldots, 2^J - 1. \tag{17}$$

Requiring that $f(x)$ be $(n + 1)$-times continuously differentiable function, it can be approximated by an n polynomial, as in (11), whose coefficients, say α_{2i}, are determined using $f(x)$. Consequently, the right-hand side of system (17), say r_l, takes the form

$$r_l = 2^{J/2} \sum_{i=0}^{n} 2^{-(i+1)J} \alpha_{2i} M_l^i(2^J) \qquad l = 2-L, 3-L, \ldots, 2^J - 1. \tag{18}$$

For $u(x,t)$ to satisfy conditions (3) and (4), the expansion coefficients $u_{J,k}(t)$ must satisfy the following relations:

$$\sum_{k=2-L}^{2^J-1} u_{J,k}(t)\varphi(2^J - k) = 2^{-J/2} g(t) \tag{19}$$

$$\sum_{k=2-L}^{2^J-1} u_{J,k}(t)[\varphi^{(1)}(2^J\beta - k) - \varphi^{(1)}(-k)] = 2^{-3J/2} \frac{\mathrm{d}}{\mathrm{d}t} m(t). \tag{20}$$

Inserting the two relations (19) and (20) into each of the two systems (13) and (18), one may arrange the results in matrix form as

$$A\dot{U} = BU + E \tag{21}$$

and

$$CU_\circ = R \tag{22}$$

where

$$U = (u_{J,2-L}(t)\, u_{J,3-L}(t) \cdots u_{J,2^J-1}(t))^T \tag{23}$$

(T denotes the transpose), $\dot{U} = \frac{\mathrm{d}}{\mathrm{d}t} U$ and $U_\circ = U|_{t=0}$. A, B and C are square matrices of size $2^J + L - 2$ such that

$$A = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ a_{3-L,2-L} & a_{3-L,3-L} & \cdots & a_{3-L,2^J-1} \\ a_{4-L,2-L} & a_{4-L,3-L} & \cdots & a_{4-L,2^J-1} \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ a_{2^J-2,2-L} & a_{2^J-2,3-L} & \cdots & a_{2^J-2,2^J-1} \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} \begin{bmatrix} \varphi^{(1)}(2^J\beta + L - 2) - \\ \varphi^{(1)}(L-2) \end{bmatrix} & \cdots & \begin{bmatrix} \varphi^{(1)}(2^J(\beta-1)+1) - \\ \varphi^{(1)}(1-2^J) \end{bmatrix} \\ b_{3-L,2-L} & \cdots & b_{3-L,2^J-1} \\ b_{4-L,2-L} & \cdots & b_{4-L,2^J-1} \\ \vdots & & \vdots \\ \vdots & \cdots & \vdots \\ \vdots & & \vdots \\ b_{2^J-2,2-L} & \cdots & b_{2^J-2,2^J-1} \\ \varphi(2^J + L - 2) & \cdots & \varphi(1) \end{pmatrix}$$

and

$$C = \begin{pmatrix} \begin{bmatrix} \varphi^{(1)}(2^J\beta + L - 2)- \\ \varphi^{(1)}(L-2) \end{bmatrix} & \cdots & \begin{bmatrix} \varphi^{(1)}(2^J(\beta-1)+1)- \\ \varphi^{(1)}(1-2^J) \end{bmatrix} \\ a_{3-L,2-L} & \cdots & a_{3-L,2^J-1} \\ a_{4-L,2-L} & \cdots & a_{4-L,2^J-1} \\ \cdot & & \cdot \\ \cdot & \cdots & \cdot \\ \cdot & & \cdot \\ a_{2^J-2,2-L} & \cdots & a_{2^J-2,2^J-1} \\ \varphi(2^J + L - 2) & \cdots & \varphi(1) \end{pmatrix}.$$

E and R are vectors given by

$$E = \left(-\sqrt{2^{-3J}}\frac{\mathrm{d}}{\mathrm{d}t}m(t) \quad \sqrt{2^J}e_{3-L} \quad \sqrt{2^J}e_{4-L} \quad \cdots \quad \sqrt{2^J}e_{2^J-2} \quad -\sqrt{2^J}g(t) \right)^T$$

and

$$R = \left(\sqrt{2^{-3J}}\frac{\mathrm{d}}{\mathrm{d}t}m(0) \quad \sqrt{2^J}r_{3-L} \quad \sqrt{2^J}r_{4-L} \quad \cdots \quad \sqrt{2^J}r_{2^J-1} \quad \sqrt{2^J}g(0) \right)^T$$

respectively.

Having obtained the system of differential equations (21) and the system of initial conditions (22) for $u_{J,k}(t)$, we are in a position to utilize a numerical integration scheme and find the expansion coefficients $u_{J,k}(t)$.

4. Numerical experiments

In actual computation, we use Daubechies 6, i.e. $L = 6$, and discretize the variable t. Let N be a natural number, $\Delta t = \frac{T}{N}$ and $t_i = i\Delta t; i = 0, 1, \ldots, N$. Define the vector U_i to be the vector given by (23) evaluated at $t = t_i$. Using this notation, we approximate $\dot{U}$ by $\frac{1}{\Delta t}(U_{i+1} - U_i)$ and U by $\frac{1}{2}(U_{i+1} + U_i)$. The system (21), then, yields the following relations:

$$\left(A - \frac{\Delta t}{2}B\right)U_{i+1} = \Delta t \cdot E + \left(A + \frac{\Delta t}{2}B\right)U_i \qquad i = 0, 1, \ldots, N$$

which with system (22) gives an approximation to the expansion coefficients $u_{J,k}(t_i)$.

In the following we consider an approximate solution $u(x, t)$ of (1) with three different sets of functions. The absolute relative errors are computed at various values of t and x. For all examples the wavelet-Galerkin approximate solution is computed at J (scaling level) $= 7$, $\Delta t = 0.001$ and each of $S(x, t)$ and $f(x)$ is interpolated by polynomials of degree 10. In the first example $\beta = 0.25$, in the second $\beta = 0.75$ and in the third $\beta = 0.5$.

Example 1. Consider $S(x, t) = -(1 + t^2)\,\mathrm{e}^x/(1 + t^2)^2$, $f(x) = \mathrm{e}^x$, $g(t) = \mathrm{e}/(1 + t^2)$ and $m(t) = (\mathrm{e}^{0.25} - 1)/(1 + t^2)$. The exact solution $u_{ex}(x, t)$ is $\mathrm{e}^x/(1 + t^2)$, see table 1.

Example 2. Let $S(x, t) = (1 + \pi^2 + t)\cos x$, $f(x) = \pi^2 \cos x$, $g(t) = (\pi^2 + t)\cos(1.0)$ and $m(t) = (\pi^2 + t)\sin(0.75)$, then the exact solution $u_{ex}(x, t)$ is $(\pi^2 + t)\cos x$, see table 2.

Table 1. Results for example 1.

x	t	Exact solution	Approximate solution	Absolute relative error
0.25	0.025	1.283 223	1.283 193	2.38×10^{-5}
	0.05	1.280 823	1.280 765	4.56×10^{-5}
	0.1	1.271 312	1.271 207	8.29×10^{-5}
0.5	0.025	1.647 691	1.647 651	2.46×10^{-5}
	0.05	1.644 610	1.644 534	4.61×10^{-5}
	0.1	1.632 397	1.632 272	7.70×10^{-5}
0.75	0.025	2.115 678	2.115 633	2.12×10^{-5}
	0.05	2.111 721	2.111 648	3.45×10^{-5}
	0.1	2.096 040	2.095 934	5.02×10^{-5}

Table 2. Results for example 2.

x	t	Exact solution	Approximate solution	Absolute relative error
0.25	0.025	9.577 005	9.587 025	2.07×10^{-6}
	0.05	9.611 228	9.611 272	4.63×10^{-6}
	0.1	9.659 674	9.659 767	9.79×10^{-6}
0.5	0.025	8.683 332	8.683 344	1.79×10^{-6}
	0.05	8.705 272	8.705 298	3.00×10^{-6}
	0.1	8.749 151	8.749 208	6.54×10^{-6}
0.75	0.025	7.239 772	7.239 781	1.21×10^{-6}
	0.05	7.258 064	7.258 080	2.24×10^{-6}
	0.1	7.294 649	7.294 680	4.29×10^{-6}

Table 3. Results for example 3.

x	t	Exact solution	Approximate solution	Absolute relative error
0.25	0.025	1.029 619	1.029 611	8.477×10^{-6}
	0.05	1.047 319	1.047 302	1.63×10^{-5}
	0.1	1.081 805	1.081 772	3.03×10^{-5}
0.5	0.025	1.115 142	1.115 134	6.46×10^{-6}
	0.05	1.131 402	1.131 388	1.23×10^{-5}
	0.1	1.163 151	1.163 125	2.20×10^{-5}
0.75	0.025	1.193 922	1.193 917	4.55×10^{-6}
	0.05	1.208 960	1.208 951	7.90×10^{-6}
	0.1	1.238 374	1.238 358	1.29×10^{-5}

Example 3. Choose $S(x,t) = 2(x + 2t + 3)/(x + 2t + 2.5)^2$, $f(x) = \ln(x + 2.5)$, $g(t) = \ln(2t + 3.5)$ and $m(t) = (2t + 3)\ln(2t + 3) - (2t + 2.5)\ln(2t + 2.5) - 0.5$, so that the exact solution $u_{ex}(x,t)$ is $\ln(x + 2t + 2.5)$, see table 3.

Remark. In the third example $S(x,t)$ is not a separable function in variables x and t. Therefore, when the variable t is discretized, then each of $S(x,t_i), i = 0, 1, \ldots, N$, is interpolated by a polynomial (assuming that $S(x,\cdot)$ is continuously differentiable).

In order to demonstrate the effect of the scaling level J and time step size Δt on the obtained approximate solutions, several numerical experiments are carried out using different values of J and Δt. It has been found that the obtained solutions for all the considered problems coincide in their convergence behaviour over the spatial variable x. Therefore,

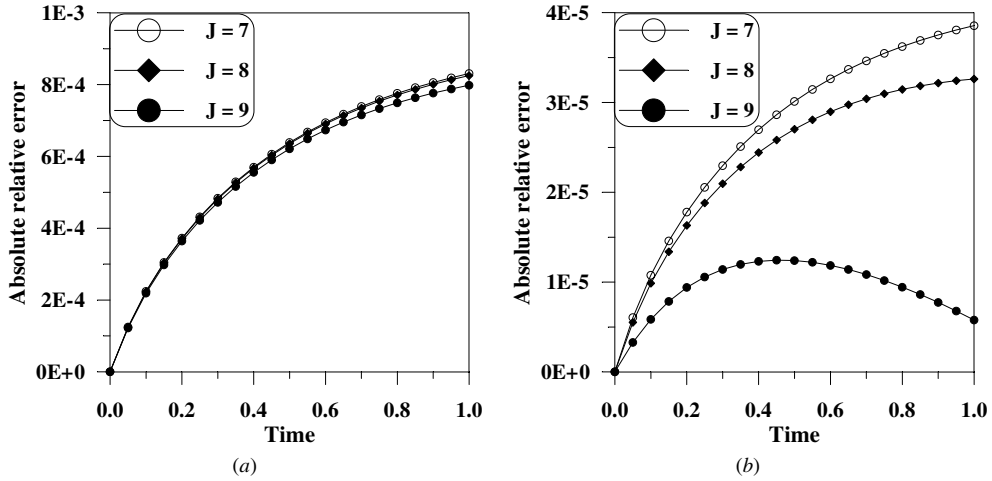

Figure 1. Absolute relative error for different values of scaling level J. (*a*) Time step size $\Delta t = 0.01$; (*b*) time step size $\Delta t = 0.0005$.

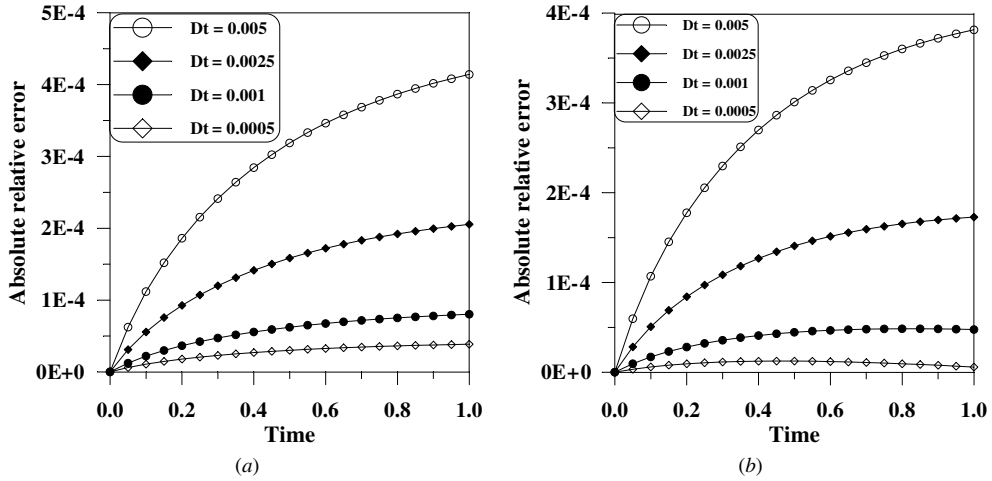

Figure 2. Absolute relative error for different values of Δt ($\mathrm{D}t \equiv \Delta t$). (*a*) Scaling level $j = 7$; (*b*) scaling level $j = 9$.

we present convergence curves to indicate the effect of J and Δt on the solution using example 3 at $x = 0.5$. The error as a function of scaling level J and the error as a function of time step Δt are displayed in figures 1 and 2, respectively. Figure 1 shows absolute relative errors for three different scaling levels ($J = 7$, $J = 8$ and $J = 9$). The absolute relative errors are computed with different values of Δt, in figure 1(*a*) $\Delta t = 0.01$, and in figure 1(*b*) $\Delta t = 0.0005$. Figure 1 declares that as J is increased, for a given Δt, the accuracy of the solution is improved. For a relatively large Δt the effect of the scaling level J on improving the accuracy is insignificant. Figure 2 displays the absolute relative errors for different values of Δt with fixed scaling level $J = 7$ in figure 2(*a*), and $J = 9$ in figure 2(*b*). It can be seen from both figures that as Δt is decreased the accuracy of the solution is improved. However, unlike the convergence of the

Galerkin procedure of [1], the proposed wavelet-Galerkin method maintains convergence for large Δt.

The present method competes well with alternative methods. Example 1 is solved in [4] using a backward difference numerical scheme for some different grid sizes. It is reported in [4], that the maximum absolute error detected for the best choice of grid sizes ($\Delta x = \Delta t = 0.0001$) is 0.006. However, for the same problem the proposed approach reveals that the maximum absolute error using $J = 7$ and $\Delta t = 0.001$ is 0.0002, while using the same J and $\Delta t = 0.0001$, it is 0.000 018.

5. Conclusion

This paper provides a technical description of the application of the wavelet-Galerkin method for the numerical solution of inhomogeneous diffusion equations subject to mass specification involving non-polynomial functions. The application of the method was shown in all essential technical details. The efficiency of the proposed method is indicated in some numerical experiments. The obtained approximate solutions are in good agreement with the exact ones. Besides, from these promising experimental results, we believe that the interest of the scheme lies in its ease of programming and implementation on such kinds of partial differential equations.

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